Asymptotic behavior of multi-scale McKean-Vlasov SDEs

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Outline

- **Backgroud**
- **Averaging principle**
- **Central limit type theorem**
- **Diffusion approximation**

1. Backgroud

Many multi-scale (also called slow-fast or two-time scales) system arise from material sciences, chemistry, fluids dynamics, biology and other application areas, such as

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1. Backgroud

Many multi-scale (also called slow-fast or two-time scales) system arise from material sciences, chemistry, fluids dynamics, biology and other application areas, such as

- In climate models, where climate-weather interactions may be studied within the averaging framework, climate being the slow motion and weather the fast one.
- In the chemistry, the dynamics of chemical reaction networks often take place on notably different times scales, from the order of nanoseconds (10^{-9} s) to the order of several days.

Averaging principle for SDEs (By Khasminskii, 1968)

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon)dt + \sigma(X_t^\varepsilon, Y_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x \in \mathbb{R}^d, \\ dY_t^\varepsilon = \frac{1}{\varepsilon} f(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} g(X_t^\varepsilon, Y_t^\varepsilon)dW_t, \quad Y_0^\varepsilon = y \in \mathbb{R}^d. \end{cases}$$

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Assume $\exists \bar{b}(x) : \mathbb{R}^d \to \mathbb{R}^d$, positive-definite symmetric matrix $A(x) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$:

$$\left| \frac{1}{T} \int_0^T \mathbb{E} b(x, Y_t^{x,y})dt - \bar{b}(x) \right| \to 0, \quad \varepsilon \to 0;$$

$$\left| \frac{1}{T} \int_0^T \mathbb{E} \sigma(x, Y_t^{x,y}) \sigma^*(x, Y_t^{x,y})dt - A(x) \right| \to 0, \quad \varepsilon \to 0;$$

where $\{Y_t^{x,y}\}_{t\geq 0}$ is the unique solution of the frozen equation:

$$dY_t^{x,y} = f(x, Y_t^{x,y})dt + g(x, Y_t^{x,y})dW_t, \quad Y_0^{x,y} = y.$$

Averaging principle says:

$$X^{\varepsilon} \to \bar{X}, \quad \text{in weak sense},$$

as $\varepsilon \to 0$, where $\bar{X}$ is the solution of the averaged equation:

$$d\bar{X}_t = \bar{b}(\bar{X}_t)dt + \bar{\sigma}(\bar{X}_t)d\bar{W}_t, \quad X_0 = x,$$

where $\bar{\sigma}(x) := \sqrt{A(x)}$, i.e., $\bar{\sigma}(x)\bar{\sigma}(x) = A(x)$.

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If frozen equation admits a unique invariant measure μ^{x}.

- $\bar{b}(x) = \int_{\mathbb{R}^d} b(x, y)\mu^{x}(dy)$
- $A(x) = \int \sigma(x, y)\sigma(x, y)^{*}\mu^{x}(dy)$

- A.Y. Veretennikov, *Math. USSR Sborn.* 1991
- J. Golec, *SAA* 1995
- D. Liu, *CMS* 2010
- W. Liu, M. Röckner, X. Sun, Y. Xie, *JDE* 2020
- F. Wu, G. Yin, *JDE* 2020
- ......

- A.Y. Veretennikov, *Math. USSR Sborn.* 1991
- J. Golec, *SAA* 1995
- D. Liu, *CMS* 2010
- W. Liu, M. Röckner, X. Sun, Y. Xie, *JDE* 2020
- F. Wu, G. Yin, *JDE* 2020
- ......
- S. Cerrai and Freidlin, *PTRF* 2009
- H. Fu, J. Liu, *JMAA* 2011
- C.E. Bréhier, *SPA* 2012
- W. Wang, A.J. Roberts, *JDE* 2012
- B. Pei, Y. Xu, G. Yin, *Nonlinear Anal.* 2017
- Z. Dong, X. Sun, H. Xiao, J. Zhai, *JDE* 2018
- P. Gao, *JSP* 2018
- ......

- D. Givon, *SIAM JMMS* 2017
- Y. Xu, J. Duan, W. Xu, *Phys. D* 2011
- J. Xu, Y. Miao, J. Liu, *DCDS-B* 2015
- B. Pei, Y. Xu, J. L. Wu, *JMAA* 2017
- Y. Shi, H. Gao, *Nonlinear Anal.* 2022
- G. Shen, J.L. Wu, R. Xiao, X. Yin, *Stoch. Dyn.* 2022
- ......

- D. Givon, *SIAM JMMS* 2017
- Y. Xu, J. Duan, W. Xu, *Phys. D* 2011
- J. Xu, Y. Miao, J. Liu, *DCDS-B* 2015
- B. Pei, Y. Xu, J. L. Wu, *JMAA* 2017
- Y. Shi, H. Gao, *Nonlinear Anal.* 2022
- G. Shen, J.L. Wu, R. Xiao, X. Yin, *Stoch. Dyn.* 2022
- ......
- J. Bao, G. Yin, C. Yuan, *Bernoulli* 2017
- X. Sun, J. Zhai, *CPAA* 2020
- X. Sun, L. Xie, Y. Xie, *Bernoulli* 2022
- ......

- Central limit type theorem (CLTT), or Normal deviations

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t, \\ dY_t^\varepsilon = \frac{1}{\varepsilon}f(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}g(X_t^\varepsilon, Y_t^\varepsilon)dW_t. \end{cases}$$

The corresponding averaged equation is following:

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Aim: How to characterize the limit process of $\{\frac{X^\varepsilon - \bar{X}}{\varepsilon^{1/2}}\}$ in $C([0,T],\mathbb{R}^n)$?

$$\frac{X^\varepsilon - \bar{X}}{\varepsilon^{1/2}} \to Z, \quad \text{weakly in} \quad C([0,T],\mathbb{R}^n), \text{ as } \varepsilon \to 0$$

- Diffusion approximation

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}K(X_t^\varepsilon, Y_t^\varepsilon) + \sigma(X_t^\varepsilon, Y_t^\varepsilon)dW_t, \\ dY_t^\varepsilon = \frac{1}{\varepsilon}f(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}g(X_t^\varepsilon, Y_t^\varepsilon)dW_t. \end{cases}$$

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Here K satisfies the central condition, i.e.,

$$\int K(x, y)\mu^x(dy) = 0.$$

Aim: How to characterize the limit process of X^ε? i.e.,

$$X^\varepsilon \to Z, \text{ weakly in } C([0, T], \mathbb{R}^n), \text{ as } \varepsilon \to 0.$$

- R.Z. Khasminskii, *Theory Probab. Appl.* 1966
- E. Pardoux, and A.Y. Veretennikov, *AOP* 2001,2003,2005
- S. Cerrai, *J. Math. Pures Appl.* 2009
- W. Hu and C. Li, *Discrete Contin. Dyn. Syst.* 2018
- S. Bourguin, S. Gailus, and K. Spiliopoulos, *Stoch. Dyn.* 2021
- M. Röckner, and L. Xie, *CMP* 2021
- L. Xie, and L. Yang, *JDE* 2021
- ......

The main techniques:

- Khasminskii's time discretization
- Martingale problem approach
- Asymptotic expansion of the solutions of Kolmogorov equation with respect to ε
- Poisson equation

  $\cdots \ \cdots$

2. Averaging principle

- In this talk, we focus on the following multi-scale McKean-Vlasov SDEs:

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dt + \sigma(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon})dW_t^1, \\ dY_t^\varepsilon = \frac{1}{\varepsilon} f(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} g(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}, Y_t^\varepsilon)dW_t^2, \end{cases} \quad (1)$$

where $\mathscr{L}_{X_t^\varepsilon}$ is the law of X_t^ε.

Theorem 1 (M. Röckner, S. and Y. Xie, AIHP, 2021)
Under some proper assumptions. We have

$$\sup_{t\in[0,T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^2 \leq C\varepsilon. \tag{2}$$

Here $\bar{X}$ is the solution of the following averaged equation,

$$\begin{cases} d\bar{X}_t = \bar{b}(\bar{X}_t, \mathscr{L}_{\bar{X}_t})dt + \sigma(\bar{X}_t, \mathscr{L}_{\bar{X}_t})dW_t^1, \\ \bar{X}_0 = x, \end{cases} \tag{3}$$

where $\bar{b}(x,\mu) = \int_{\mathbb{R}^m} b(x,\mu,y)\nu^{x,\mu}(dy)$ and $\nu^{x,\mu}$ denotes the unique invariant measure for the transition semigroup of the frozen equation.

♣ Idea of Proof of Averaging principle

For simplification, $\sigma(x,\mu)\equiv\sigma$. Recall that

$$X_t^\varepsilon = X_0 + \int_0^t b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon)ds + \sigma W_t^1,$$
$$\bar{X}_t = X_0 + \int_0^t \bar{b}(\bar{X}_s, \mathscr{L}_{\bar{X}_s})ds + \sigma W_t^1.$$

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$$\bar{X}_t = X_0 + \int_0^t \bar{b}(\bar{X}_s, \mathscr{L}_{\bar{X}_s})ds + \sigma W_t^1.$$

Thus

$$\begin{aligned} X_t^\varepsilon - \bar{X}_t = & \int_0^t \left[b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - \bar{b}(\bar{X}_s, \mathscr{L}_{\bar{X}_s})\right] ds \\ = & \int_0^t \left[b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - \bar{b}(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon})\right] ds \\ & + \int_0^t \left[\bar{b}(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}) - \bar{b}(\bar{X}_s, \mathscr{L}_{\bar{X}_s})\right] ds. \end{aligned}$$

Note that $\bar{b}$ is Lipschitz continuity, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^2 \\
\leq\ & C \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - \bar{b}(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon})ds\right|^2 \\
&+ C_T \int_0^T \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^2 dt.
\end{aligned}
$$

By Gronwall's inequality, we get

$$
\begin{aligned}
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\end{aligned}
$$

Now, consider the following Poisson equation with parameter x:

$$-\mathscr{L}_2(x,\mu)\Phi(x,\mu,\cdot)(y) = b(x,\mu,y) - \bar{b}(x,\mu), \quad y \in \mathbb{R}^{d_2}, \tag{4}$$

where $\mathscr{L}_2(x,\mu)$ is the generator of the frozen equation:

$$\begin{cases} dY_s^{x,\mu,y} = f(x,\mu,Y_s^{x,\mu,y})ds + dW_s^2, \\ Y_0^{x,\mu,y} = y. \end{cases} \tag{5}$$

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Poisson equation (4) admits a unique solution

$$\Phi(x,\mu,y) := \int_0^\infty \left[\mathbb{E}b(x,\mu,Y_s^{x,\mu,y}) - \bar{b}(x,\mu)\right] ds.$$

By Itô's formula, we have

$$\begin{aligned}
\Phi(X_t^\varepsilon, \mathscr{L}_{X_t^\varepsilon}, Y_t^\varepsilon) &= \Phi(X_0, \mathscr{L}_{X_0}, Y_0) \\
&+ \int_0^t \mathbb{E}\left[b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon)\partial_\mu \Phi(x, \mathscr{L}_{X_s^\varepsilon}, y)(X_s^\varepsilon)\right]|_{x=X_s^\varepsilon, y=Y_s^\varepsilon}\, ds \\
&+ \int_0^t \frac{1}{2}\mathbb{E}\mathrm{Tr}\left[\sigma\sigma^* \partial_z \partial_\mu \Phi(x, \mathscr{L}_{X_s^\varepsilon}, y)(X_s^\varepsilon)\right]|_{x=X_s^\varepsilon, y=Y_s^\varepsilon}\, ds \\
&+ \int_0^t \mathscr{L}_1(\mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon)\Phi(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) ds \\
&+ \frac{1}{\varepsilon}\int_0^t \mathscr{L}_2(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon})\Phi(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) ds + M_t^{\varepsilon,1} + \frac{1}{\sqrt{\varepsilon}} M_t^{\varepsilon,2},
\end{aligned}$$

where $M_t^{\varepsilon,1}$ and $M_t^{\varepsilon,2}$ are two local martingales,

$$M_t^{\varepsilon,1} := \int_0^t \partial_x \Phi(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}) \cdot \sigma dW_s^1;$$

$$M_t^{\varepsilon,2} := \int_0^t \partial_y \Phi(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}) \cdot g(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) dW_s^2.$$

Then we have

$$\sup_{t\in[0,T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^2 \leq \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - \bar{b}(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}) ds\right|^2$$

$$= \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t \mathscr{L}_2(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}) \Phi(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) ds\right|^2$$

$$\leq \varepsilon^2 \sup_{t\in[0,T]} \mathbb{E}\,|A_t^\varepsilon|^2 + \varepsilon^2 \sup_{t\in[0,T]} \mathbb{E}\left|M_t^{\varepsilon,1}\right|^2 + \varepsilon \sup_{t\in[0,T]} \mathbb{E}\left|M_t^{\varepsilon,2}\right|^2.$$

3. Central Limit Type Theorem

Theorem 2 (W. Hong, S. Li, W. Liu and S., arxiv, 2021+)
Under some proper assumptions. $\{Z^{\varepsilon} := \frac{X^{\varepsilon}-\bar{X}}{\sqrt{\varepsilon}}\}_{\varepsilon>0}$ converges weakly in $C([0,T];\mathbb{R}^n)$ to the solution of following equation as $\varepsilon \to 0$,

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$$
\begin{aligned}
dZ_t = \; & \partial_x \bar{b}(\bar{X}_t, \mathscr{L}_{\bar{X}_t}) \cdot Z_t dt + \mathbb{E}\left[\partial_\mu \bar{b}(u, \mathscr{L}_{\bar{X}_t})(\bar{X}_t) \cdot Z_t\right]\Big|_{u=\bar{X}_t} dt \\
& + \left[\partial_x \sigma(\bar{X}_t, \mathscr{L}_{\bar{X}_t}) \cdot Z_t\right] dW_t^1 + \mathbb{E}\left[\partial_\mu \sigma(u, \mathscr{L}_{\bar{X}_t})(\bar{X}_t) \cdot Z_t\right]\Big|_{u=\bar{X}_t} dW_t^1 \\
& + \Theta(\bar{X}_t, \mathscr{L}_{\bar{X}_t}) dW_t, \; Z_0 = 0,
\end{aligned} \tag{6}
$$

where W_t is a $\mathbb{R}^n$-dimensional standard Brownian motion independent of W_t^1 on probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and

$$
\Theta(x,\mu) := \left(\overline{(\partial_y \Phi_g)(\partial_y \Phi_g)^*}\right)^{\frac{1}{2}}(x,\mu).
$$

♣ Idea of Proof of CLTT

For simplification, $\sigma(x,\mu)\equiv\sigma$. $Z_t^\varepsilon=\frac{X_t^\varepsilon-X_t}{\sqrt{\varepsilon}}$ satisfies

$$dZ_t^\varepsilon=\frac{1}{\sqrt{\varepsilon}}\left[b(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon},Y_t^\varepsilon)-\bar{b}(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon})\right]dt$$

$$+\frac{1}{\sqrt{\varepsilon}}\left[\bar{b}(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon})-\bar{b}(\bar{X}_t,\mathscr{L}_{X_t^\varepsilon})\right]dt+\frac{1}{\sqrt{\varepsilon}}\left[\bar{b}(\bar{X}_t,\mathscr{L}_{X_t^\varepsilon})-\bar{b}(\bar{X}_t,\mathscr{L}_{\bar{X}_t})\right]dt.$$

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$$+\frac{1}{\sqrt{\varepsilon}}\Big[\bar{b}(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon})-\bar{b}(\bar{X}_t,\mathscr{L}_{X_t^\varepsilon})\Big]dt+\frac{1}{\sqrt{\varepsilon}}\Big[\bar{b}(\bar{X}_t,\mathscr{L}_{X_t^\varepsilon})-\bar{b}(\bar{X}_t,\mathscr{L}_{\bar{X}_t})\Big]dt.$$

We introduce an auxiliary process η_t^ε,

$$d\eta_t^\varepsilon=\frac{1}{\sqrt{\varepsilon}}\Big[b(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon},Y_t^\varepsilon)-\bar{b}(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon})\Big]dt$$
$$+\partial_x\bar{b}(\bar{X}_t,\mathscr{L}_{\bar{X}_t})\cdot\eta_t^\varepsilon dt+\mathbb{E}\Big[\partial_\mu\bar{b}(u,\mathscr{L}_{\bar{X}_t})(\bar{X}_t)\cdot\eta_t^\varepsilon\Big]\Big|_{u=\bar{X}_t}dt$$

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Step 2: Note that $\eta^\varepsilon(t) = I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t)$, where

$$I_1^\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}}\int_0^t \left[b(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - \bar{b}(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon})\right] ds,$$

$$I_2^\varepsilon(t) := \int_0^t \partial_x \bar{b}(\bar{X}_s, \mathscr{L}_{\bar{X}_s}) \cdot \eta_s^\varepsilon ds,$$

$$I_3^\varepsilon(t) := \int_0^t \mathbb{E}\left[\partial_\mu \bar{b}(u, \mathscr{L}_{\bar{X}_s})(\bar{X}_s) \cdot \eta_s^\varepsilon\right]\Big|_{u=\bar{X}_s} ds.$$

Proving $\Pi^\varepsilon := (\eta^\varepsilon, I_1^\varepsilon, I_2^\varepsilon, I_3^\varepsilon)$, is tight in $C([0,T];\mathbb{R}^{4n})$.

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Step 2: Note that $\eta^\varepsilon(t)=I_1^\varepsilon(t)+I_2^\varepsilon(t)+I_3^\varepsilon(t)$, where

$$I_1^\varepsilon(t):=\frac{1}{\sqrt{\varepsilon}}\int_0^t\left[b(X_s^\varepsilon,\mathscr{L}_{X_s^\varepsilon},Y_s^\varepsilon)-\bar{b}(X_s^\varepsilon,\mathscr{L}_{X_s^\varepsilon})\right]ds,$$

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$$I_3^\varepsilon(t):=\int_0^t\mathbb{E}\left[\partial_\mu\bar{b}(u,\mathscr{L}_{\bar{X}_s})(\bar{X}_s)\cdot\eta_s^\varepsilon\right]\Big|_{u=\bar{X}_s}ds.$$

Proving $\Pi^\varepsilon:=(\eta^\varepsilon,I_1^\varepsilon,I_2^\varepsilon,I_3^\varepsilon)$, is tight in $C([0,T];\mathbb{R}^{4n})$.

Step 3: Identify the weak limit of I_1^ε, I_2^ε and I_3^ε in $C([0,T];\mathbb{R}^n)$.

Key: Skorohod representation theorem.

Recall that $\Phi(x,\mu,y)$ be the unique solution of the Possion equation

$$-\mathscr{L}_2(x,\mu)\Phi(x,\mu,\cdot)(y)=b(x,\mu,y)-\bar{b}(x,\mu).$$

By Itô's formula,

$$\begin{aligned}\Phi(X_t^\varepsilon,\mathscr{L}_{X_t^\varepsilon},Y_t^\varepsilon)&=\Phi(X_0,\mathscr{L}_{X_0},Y_0)\\&+\int_0^t\mathbb{E}\left[b(X_s^\varepsilon,\mathscr{L}_{X_s^\varepsilon},Y_s^\varepsilon)\partial_\mu\Phi(x,\mathscr{L}_{X_s^\varepsilon},y)(X_s^\varepsilon)\right]|_{x=X_s^\varepsilon,y=Y_s^\varepsilon}\,ds\\&+\int_0^t\frac{1}{2}\mathbb{E}\mathrm{Tr}\left[\sigma\sigma^*\partial_z\partial_\mu\Phi(x,\mathscr{L}_{X_s^\varepsilon},y)(X_s^\varepsilon)\right]|_{x=X_s^\varepsilon,y=Y_s^\varepsilon}\,ds\\&+\int_0^t\mathscr{L}_1(\mathscr{L}_{X_s^\varepsilon},Y_s^\varepsilon)\Phi(X_s^\varepsilon,\mathscr{L}_{X_s^\varepsilon},Y_s^\varepsilon)ds\\&+\frac{1}{\varepsilon}\int_0^t b(X_s^\varepsilon,\mathscr{L}_{X_s^\varepsilon},Y_s^\varepsilon)-\bar{b}(X_s^\varepsilon,\mathscr{L}_{X_s^\varepsilon})ds+M_t^{1,\varepsilon}+\frac{1}{\sqrt{\varepsilon}}M_t^{2,\varepsilon},\end{aligned}$$

where $M_t^{1,\varepsilon}, M_t^{2,\varepsilon}$ are two local martingales,

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$$M_t^{2,\varepsilon} = \int_0^t \partial_y \Phi(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) \cdot g(X_s^\varepsilon, \mathscr{L}_{X_s^\varepsilon}, Y_s^\varepsilon) dW_s^2.$$

Then it is easy to see

$$I_1^\varepsilon(t) = \sqrt{\varepsilon} B_t^\varepsilon + M_t^{2,\varepsilon}. \tag{7}$$

According to martingale representation theorem, $M_t^{2,\varepsilon}$ converges to

$$M_t^2 = \int_0^t \left(\overline{(\partial_y \Phi_g)(\partial_y \Phi_g)^*}\right)^{\frac{1}{2}} (\bar{X}_s, \mathscr{L}_{\bar{X}_s}) dW_s.$$

3. Diffusion Approximation

We consider the following stochastic systems

$$\begin{cases} dX^{\varepsilon}_t = b(X^{\varepsilon}_t, \mathscr{L}_{X^{\varepsilon}_t}, Y^{\varepsilon}_t)dt + \frac{1}{\sqrt{\varepsilon}} K(X^{\varepsilon}_t, \mathscr{L}_{X^{\varepsilon}_t}, Y^{\varepsilon}_t)dt + \sigma(X^{\varepsilon}_t, \mathscr{L}_{X^{\varepsilon}_t}, Y^{\varepsilon}_t)dW_t, \\ dY^{\varepsilon}_t = \frac{1}{\varepsilon} f(X^{\varepsilon}_t, \mathscr{L}_{X^{\varepsilon}_t}, Y^{\varepsilon}_t)dt + \frac{1}{\sqrt{\varepsilon}} h(X^{\varepsilon}_t, \mathscr{L}_{X^{\varepsilon}_t}, Y^{\varepsilon}_t)dt + \frac{1}{\sqrt{\varepsilon}} g(X^{\varepsilon}_t, \mathscr{L}_{X^{\varepsilon}_t}, Y^{\varepsilon}_t)dW_t, \\ X^{\varepsilon}_0 = \xi, \;\; Y^{\varepsilon}_0 = \zeta, \end{cases} \tag{8}$$

where K satisfies the centering condition, i.e.,

$$\int_{\mathbb{R}^m} K(x, \mu, y)\nu^{x,\mu}(dy) = 0.$$

- Z. Bezemek, K. Spiliopoulos, *Rate of homogenization for fully-coupled McKean-Vlasov SDEs*, Stoch. Dyn. to appear

♣ Martingale problem approach

Key idea: The limiting process is the solution of the martingale problem associated to the operator L_μ given by

$$L_\mu := \sum_{i=1}^{n} \Theta_i(x,\mu)\partial_{x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\Sigma\Sigma^*)_{ij}(x,\mu)\partial_{x_i}\partial_{x_j},$$

where

$$\Theta(x,\mu) := \overline{b + \partial_x\Phi_K + \partial_y\Phi_h + \mathrm{Tr}\left[\partial^2_{xy}\Phi_{\sigma g^*}\right]}(x,\mu),$$

$$\Sigma(x,\mu) := \left(\overline{(K\otimes\Phi) + (K\otimes\Phi)^* + (\sigma g^*)\partial_y\Phi + [(\sigma g^*)\partial_y\Phi]^*} + \overline{(\sigma\sigma^*)}\right)^{\frac{1}{2}}(x,\mu).$$

Theorem 3 (W. Hong, S. Li and S., arxiv, 2022)
Under some proper assumptions. $\{X^{\varepsilon}\}_{\varepsilon>0}$ converges weakly in $C([0,T];\mathbb{R}^n)$, as $\varepsilon \to 0$, to the solution of following equation

$$dX_t = \Theta(X_t, \mathscr{L}_{X_t})dt + \Sigma(X_t, \mathscr{L}_{X_t})d\hat{W}_t, \quad X_0 = \xi, \tag{9}$$

where $\hat{W}_t$ is a n-dimensional standard Brownian motion.

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where $\hat{W}_t$ is a n-dimensional standard Brownian motion.

- Note that there is a gap about the diffusion coefficient Σ, that is, it is unclear whether the term

$$\overline{(K\otimes\Phi) + (K\otimes\Phi)^* + (\sigma g^*)\partial_y\Phi + [(\sigma g^*)\partial_y\Phi]^* + (\sigma\sigma^*)}$$

is positive semi-definite?

♣ Martingale characterization

Theorem 4 (W. Hong, S. Li and S., arxiv, 2022)
Under some proper assumptions. $\{X^\varepsilon\}_{\varepsilon>0}$ converges weakly in $C([0,T];\mathbb{R}^n)$, as $\varepsilon \to 0$, to the solution of following equation

$$dX_t = \Theta(X_t, \mathscr{L}_{X_t})dt + \tilde{\Sigma}(X_t, \mathscr{L}_{X_t})d\tilde{W}_t,$$

where $\tilde{W}_t$ is a n-dimensional standard Brownian motion and

$$\tilde{\Sigma}(x,\mu) := \left(\overline{(\partial_y \Phi_g + \sigma)(\partial_y \Phi_g + \sigma)^*}\right)^{\frac{1}{2}}(x,\mu).$$

♣ Martingale characterization

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- It will be asserted that $\Sigma, \tilde{\Sigma}$ are essential the same.

Thank you very much!